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Bending Analysis of Deep Beam Using Refined Shear Deformation Theory

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Abstract- A trigonometric shear deformation theory for bending of deep beams, taking into account transverse shear deformation effects, is developed. The number of variables in the present theory is same as that in the first order shear deformation theory. The displacement field used in terms of thickness coordinate to represent the shear deformation effects. The noteworthy feature of this theory is that the transverse shear stresses can be obtained directly from the use of constitutive relations with excellent accuracy, satisfying the shear stress free conditions on the top and bottom surfaces of the beam. Hence, the theory obviates the need of shear correction factor. Governing differential equations and boundary conditions are obtained by using the principle of virtual work. The deep isotropic beams are considered for the numerical studies to demonstrate the efficiency of the theory. It has been shown that the theory is capable of predicting the local effect of stress concentration due to fixity of support. The cantilever isotropic beam subjected to uniformly varying load is examined using the present theory. Results obtained are discussed critically with those of other refined theories.

Keywords Deep beam, trigonometric shear deformation, principle of virtual work, equilibrium equations, displacement, stress

1. Introduction

It is well-known that elementary theory of bending of beam based on Euler-Bernoulli hypothesis disregards the effects of the shear deformation and stress concentration. The theory is suitable for slender beams and is not suitable for deep or thick beams since it is based on the assumption that the transverse normal to neutral axis remains so during bending and after bending, implying that the transverse shear strain is zero. Since theory neglects the transverse shear deformation, it underestimates deflections in case of deep beams where shear deformation effects are significant.

Bresse [5], Rayleigh [16] and Timoshenko [20] were the pioneer investigators to include refined effects such as rotatory inertia and shear deformation in the beam theory. Timoshenko showed that the effect of transverse vibration of prismatic bars. This theory is now widely referred to as Timoshenko beam theory or first order shear deformation theory (FSDT) in the literature. In this theory transverse shear strain distribution is assumed to be constant through the beam thickness and thus requires shear correction factor to appropriately represent the strain energy of deformation. Cowper [6] has given refined expression for the shear correction factor for different cross-sections of beam. The accuracy of Timoshenko beam theory for transverse vibrations of simply supported beam in respect of the fundamental frequency is verified by Cowper [7] with a plane stress exact elasticity solution. To remove the discrepancies in classica11 and first order shear deformation theories, higher order or refined shear deformation 2theories were developed and are available in the open literature for static and vibration analysis of beam.
Levinson [15], Bickford [4], Rehfield and Murty [18], Krishna Murty [14], Baluch et al. [2], Bhimaraddi and Chandrashekhara [3] presented parabolic shear deformation theories assuming a higher variation of axial displacement in terms of thickness coordinate. These theories satisfy shear stress free boundary conditions on top and bottom surfaces of beam and thus obviate the need of shear correction factor. Irretier [12] studied the refined dynamical effects in linear, homogenous beam according to theories, which exceed the limits of the Euler-Bernoulli beam theory. These effects are rotary inertia, shear deformation, rotary inertia and shear deformation, axial pre-stress, twist and coupling between bending and torsion.

Kant and Gupta [13], Heyliger and Reddy [10] presented finite element models based on higher order shear deformation uniform rectangular beams. However, these displacement based finite element models are not free from phenomenon of shear locking (Averill and Reddy [1]; Reddy [17]).

There is another class of refined theories, which includes trigonometric functions to represent the shear deformation effects through the thickness. Stein [19] developed refined shear deformation theories for deep beams including sinusoidal function in terms of thickness coordinate in displacement field. However, with these theories shear stress free boundary conditions are not satisfied at top and bottom surfaces of the beam. A study of literature by Ghugal and Shimpi [8] indicates that the research work dealing with flexural analysis of deep beams using refined trigonometric and hyperbolic shear deformation theories is very scarce and is still in infancy. Ghugal and Dahake [22,23] presented trigonometric shear deformation theory using sinusoidal function in thickness coordinates In this paper development of theory and its application to deep fixed beams is presented.

2. Development Of Theory

The beam under consideration as shown in Figure1 occupies in $0-x-y-z$ Cartesian coordinate system the region:

$$0 \le x \le L \; ; \quad 0 \le y \le b \; ; \quad -\frac{h}{2} \le z \le \frac{h}{2}$$

where x, y, z are Cartesian coordinates, L and b are the length and width of beam in the x and y directions respectively, and h is the thickness of the beam in the z-direction. The beam is made up of homogeneous, linearly elastic isotropic material.

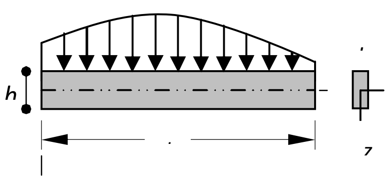

Fig. 1. Beam under bending in x-z plane.

2.1 The displacement field

The displacement field of the present beam theory is of the form:

$$u(x,z) = -z\frac{dw}{dx} + z\left(1-\frac{4}{3}\frac{z^2}{h^2}\right)\phi$$

$$w(x,z) = w(x) \tag{1}$$

where u is the axial displacement in x direction and w is the transverse displacement in z direction of the beam. The sinusoidal function is assigned according to the shear stress distribution through the thickness of the beam. The function ϕ represents rotation of the beam at neutral axis, which is an unknown function to be determined. The normal and shear strains obtained within the framework of linear theory of elasticity using displacement field given by Equation (1) are as follows.

Normal strain:

$$\varepsilon_x = \frac{du}{dx} = -\frac{d^2w}{dx^2} + z\left(1-\frac{4}{3}\frac{z^2}{h^2}\right)\frac{d\phi}{dx} \tag{2}$$

Shear strain:

$$\gamma_{zx} = \frac{du}{dz} + \frac{dw}{dx} \tag{3}$$

The stress-strain relationships used are as follows:

$$\sigma_x = E\varepsilon_x, \; \tau_{zx} = G\gamma_{zx} \tag{4}$$

2.2 Governing equations and boundary conditions

Using the expressions for strains and stresses (2) through (4) and using the principle of virtual work, variationally consistent governing differential equations and boundary conditions for the beam under consideration can be obtained. The principle of virtual work when applied to the beam leads to:

$$b\int_{x=0}^{x=L}\int_{z=-h/2}^{z=+h/2}\left(\sigma_x\delta\varepsilon_x + \tau_{zx}\delta\gamma_{zx}\right)dx\,dz - \int_{x=0}^{x=L} q(x)\,\delta w\,dx = 0 \tag{5}$$

where the symbol δ denotes the variational operator. Employing Greenôs theorem in Eqn. (4) successively, we obtain the coupled Euler-Lagrange equations which are the governing differential equations and associated boundary conditions of the beam. The governing differential equations obtained are as follows:

$$EI\frac{d^4w}{dx^4} - \frac{4}{5}EI\frac{d^3\phi}{dx^3} - q(x) = 0 \tag{6}$$

$$\frac{4}{5}EI\frac{d^3w}{dx^3} - \frac{68}{105}EI\frac{d^2\phi}{dx^2} + \frac{8}{15}GA\phi = 0 \tag{7}$$

The associated consistent natural boundary conditions obtained are of following form:
At the ends x = 0 and x = L

$$V_x = EI\frac{d^3w}{dx^3} - \frac{4}{5}EI\frac{d^3\phi}{dx^3} = 0 \quad \text{or } w \text{ is prescribed} \tag{8}$$

$$M_x = EI\frac{d^2w}{dx^2} - \frac{4}{5}EI\frac{d\phi}{dx} = 0 \quad \text{or } \frac{dw}{dx} \text{ is prescribed} \tag{9}$$

$$M_x = \frac{4}{5}EI\frac{d^2w}{dx^2} - \frac{68}{105}EI\frac{d\phi}{dx} \quad \text{or } \phi \text{ is prescribed} \tag{10}$$

Thus the boundary value problem of the beam bending is given by the above variationally consistent governing differential equations and boundary conditions.

2.3 The general solution of governing equilibrium equations of the beam

The general solution for transverse displacement w(x) and warping function ϕ(x) is obtained using Eqns. (6) and (7) using method of solution of linear differential equations with constant coefficients. Integrating and rearranging the first governing Eqn. (6), we obtain the following equation

$$\frac{d^3w}{dx^3} = \frac{4}{5}\frac{d^2\phi}{dx^2} + \frac{Q(x)}{EI}$$

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(11)

where Q(x) is the generalized shear force for beam and it is given by

$$Q(x) = \int_0^x q\,dx + C_1$$

Now the second governing Eqn (7) is rearranged in the following form

$$\frac{d^3 w}{dx^3} = \frac{\pi}{4}\frac{d^2\phi}{dx^2} - \beta\,\phi$$

(12)

A single equation in terms of ϕ is now obtained using Eqns (11) and (12) as:

$$\frac{d^2\phi}{dx^2} - \lambda^2\phi = \frac{Q(x)}{\alpha EI}$$

(13)

where constants α, β and λ in Eqns. (12) and (13) are as follows

$$\alpha = \left(\frac{17}{21} - \frac{4}{5}\right), \beta = \frac{2GA}{3EI}, \lambda^2 = \frac{\beta}{\alpha}$$

The general solution of Eqn. (13) is as follows:

$$\phi(x) = C_2\cosh\lambda x + C_3\sinh\lambda x - \frac{Q(x)}{\beta EI}$$

(14)

The equation of transverse displacement w(x) is obtained by substituting the expression of ϕ(x) in Eqn. (12) and then integrating it thrice with respect to x. The general solution for w(x) is obtained as follows:

$$EIw(x) = \iiint q\,dx\,dx\,dx + \frac{c_1 x^3}{6} + \left(\frac{17\lambda^2}{21} - \beta\right)\frac{EI}{\lambda^3}\left(c_2\sinh\lambda x - c_3\cosh\lambda x\right) + c_4\frac{x^2}{2} + c_5 x + c_6$$

(15)

Where are arbitrary constants and can be obtained by imposing boundary conditions of beam

3. Illustrative Example

In order to prove the efficacy of the present theory, the following numerical example is considered. The following material properties for beam are used E = 210 GPa, ε = 0.3 and ρ = 7800 kg/m3, where E is the Youngôs modulus, ρ is the density, and ε is the Poissonôs ratio of beam material.

A cantilever beam has its origin at left hand side support and is fixed at x = 0. The beam is subjected to uniformly varying load on surface z = +h/2 acting in the downward z direction with maximum intensity of load q_0.

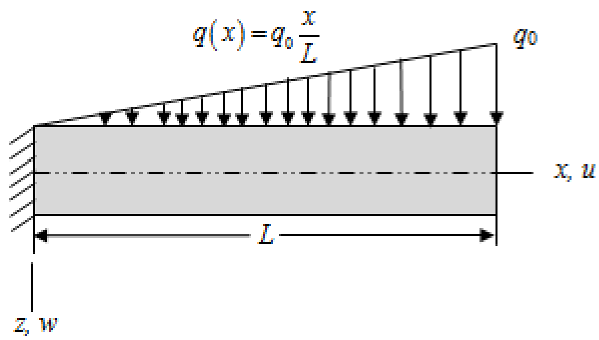


Fig. 2. Cantilever beam with uniformly varying load

General expressions obtained for $w(x)$ and $\phi(x)$ are as follows:

The axial displacement and stresses obtained based on above solutions are as follows

$$w(x) = \frac{q_0 L^4}{10Ebh^3}\left[\begin{array}{l}\frac{x^5}{L^5} + \frac{20x^2}{L^2} - \frac{10x^3}{L^3} + \frac{255}{21}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{x^2}{2L^2} - \frac{x^3}{6L^3}\right) \\ -\frac{6h^2}{L^2}\frac{E}{G}\left(\frac{\sinh\lambda x - \cosh\lambda x + 1}{\lambda L} - \frac{x}{L}\right)\end{array}\right] \quad (16)$$

$$\phi(x) = \frac{1}{15}\frac{q_0 L}{\beta EI}\left(1 + 5\frac{x^3}{L^3} + \sinh\lambda x - \cosh\lambda x\right) \quad (17)$$

$$u = \frac{q_0 h}{Eb}\left[\begin{array}{l}-\frac{1}{10}\frac{z}{h}\frac{L^3}{h^3}\left(\begin{array}{l}\frac{5x^4}{L^4} - \frac{30x^2}{L^2} + \frac{40x}{L} + \frac{255}{21}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{x}{L} - \frac{x^2}{2L^2}\right) \\ -6\frac{E}{G}\frac{h^2}{L^2}\left(\cosh\lambda x - \sinh\lambda x - 1\right)\end{array}\right) \\ +\frac{z}{h}\left(1 - \frac{4}{3}\frac{z^2}{h^2}\right)\frac{3}{4}\frac{L}{h}\left(\sinh\lambda x - \cosh\lambda x - 1 - \frac{x^2}{L^2}\right)\end{array}\right]$$

(18)

$$\sigma_x = \frac{q_0}{b}\left\{ \begin{aligned} & -\frac{z}{h}\frac{1}{10}\frac{L^2}{h^2}\left[\begin{aligned} & \frac{20x^3}{L^3} + 40 - \frac{60x}{L} + \frac{255}{21}\frac{E}{G}\frac{h^2}{L^2}\left(1-\frac{x}{L}\right) \\ & -6\frac{h^2}{L^2}\frac{E}{G}(\lambda L\sinh\lambda x - \lambda L\cosh\lambda x) \end{aligned} \right] \\ & -\frac{z}{h}\frac{E}{G}\frac{3}{4}\left(1-\frac{4}{3}\frac{Z^2}{h^2}\right)\left[(\lambda L\sinh\lambda x - \lambda L\cosh\lambda x) - \frac{2x}{L}\right] \end{aligned} \right\}$$

$$\tau_{zx}^{CR} = \frac{3}{4}\frac{q_0}{b}\frac{L}{h}\cos\left(\frac{\pi z}{h}\right)\left(\sinh\lambda x - \cosh\lambda x + 1 - \frac{x^2}{L^2}\right)$$

$$\tau_{zx}^{EE} = \frac{3}{4}\frac{L}{h}\left(\frac{4z^2}{h^2}-1\right)\left[\frac{x^2}{L^2} - 1 - \frac{51}{252}\frac{E}{G}\frac{h^2}{L^2} - \frac{1}{10}\frac{h^2}{L^2}\frac{E}{G}\lambda^2 L^2(\cosh\lambda x - \sinh\lambda x)\right]$$
$$+\frac{3}{4}\frac{E}{G}\frac{h}{L}\left(\frac{1}{2}\frac{z^2}{h^2} - \frac{1}{3}\frac{z^4}{h^4} - \frac{5}{48}\right)\left[(\lambda^2 L^2(\cosh\lambda x - \sinh\lambda x) - 2\right]$$

4 Results

The results for inplane displacement, transverse displacement, axial and transverse stresses are presented in the following non dimensional form for the purpose of presenting the results in this paper.

$$\bar{u} = \frac{E\,b\,u}{q_0 h},\ \bar{w} = \frac{10\,E\,b\,h^3 w}{q_0 L^4},\ \bar{\sigma}_x = \frac{b\,\sigma_x}{q_0},\ \bar{\tau}_{zx} = \frac{b\,\tau_{zx}}{q_0}$$

Table 1. Non-dimensional axial displacement ($\bar{u}$) at (x=0.75L, z = h/2), transverse deflection ($\bar{w}$) at (x = 0.75L, z =0.0), axial Stress ($\bar{\sigma}_x$) at (x =0, z = h/2), maximum transverse shear stresses $\bar{\tau}_{zx}^{CR}$ and $\bar{\tau}_{zx}^{EE}$ (x=0.01L, z =0.0) of the beam for aspect ratio 4 and 10

Source	Aspect ratio	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
Present TSDT		-52.00	8.40	-42.92	1.53	-2.11
Ghugal and Dahake [23]		-52.02	8.40	-42.54	1.47	-2.05
Ghugal and Sharma [9]	4	-51.99	8.40	-44.88	1.74	-3.41
Krishna Murthy [14]		-52.00	8.40	-42.93	1.53	-2.12
Timoshenko [20]		-47.06	7.59	-32.00	0.97	2.99
Bernoulli-Euler		-47.06	7.26	-32.00	—	2.99
Present TSDT		-747.7	7.45	-224.9	6.25	3.14
Ghugal and Dahake [23]		-747.7	7.45	-223.9	6.20	2.72
Ghugal and Sharma [9]	10	-747.6	7.45	-229.8	6.63	3.18
Krishna Murthy [14]		-747.7	7.45	-244.9	6.13	3.14
Timoshenko [20]		-735.3	7.32	-200.0	6.25	7.49
Bernoulli-Euler		-735.3	7.26	-200.0	—	7.49

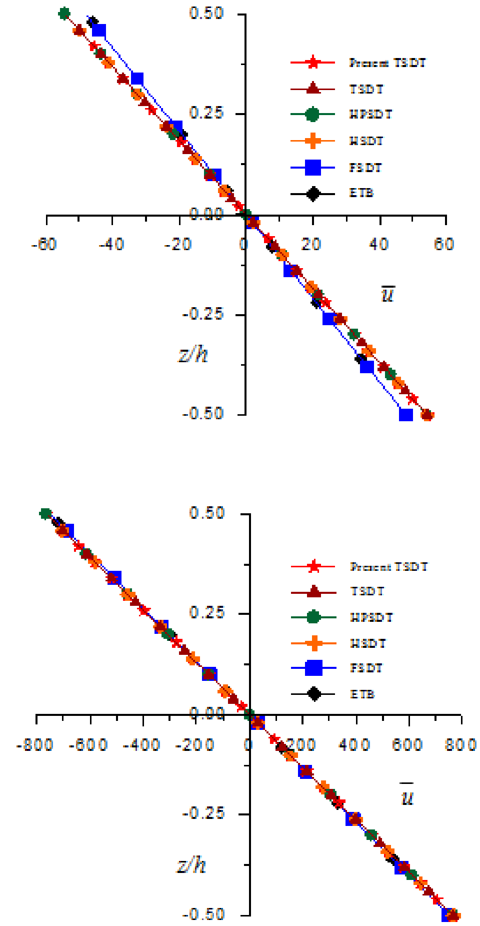
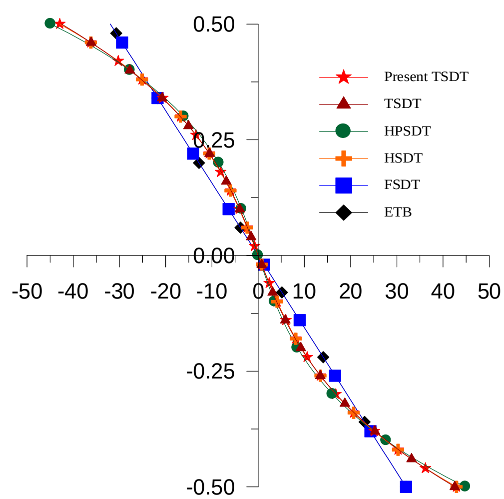


Fig. 3. Variation of axial displacement ($\bar{u}$) through the thickness of beam at (x = 0.75L, z) for aspect ratio 4 and 10.



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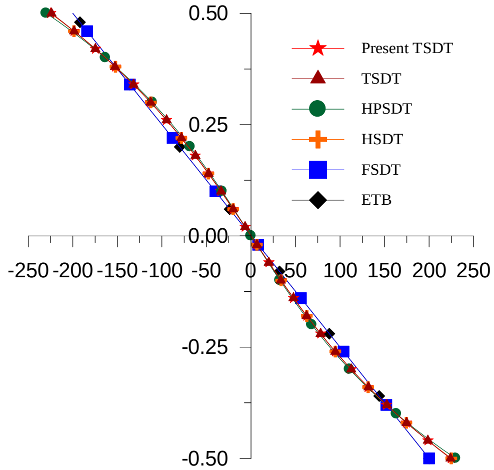

Fig. 4. Variation of axial stress $(\bar{\sigma}_x)$ through the thickness of cantilever beam at (x= 0, z) for aspect ratio 4 and 10.

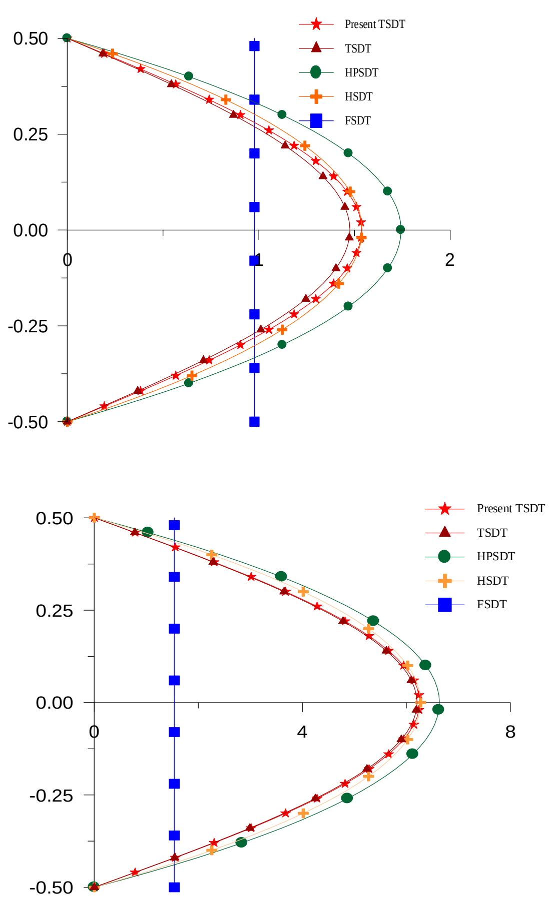

Fig. 5. Variation of transverse shear stress $(\bar{\tau}_{zx})$ through the thickness of cantilever beam at (x = 0.01L, z) obtain using constitutive relation for aspect ratio 4 and 10.

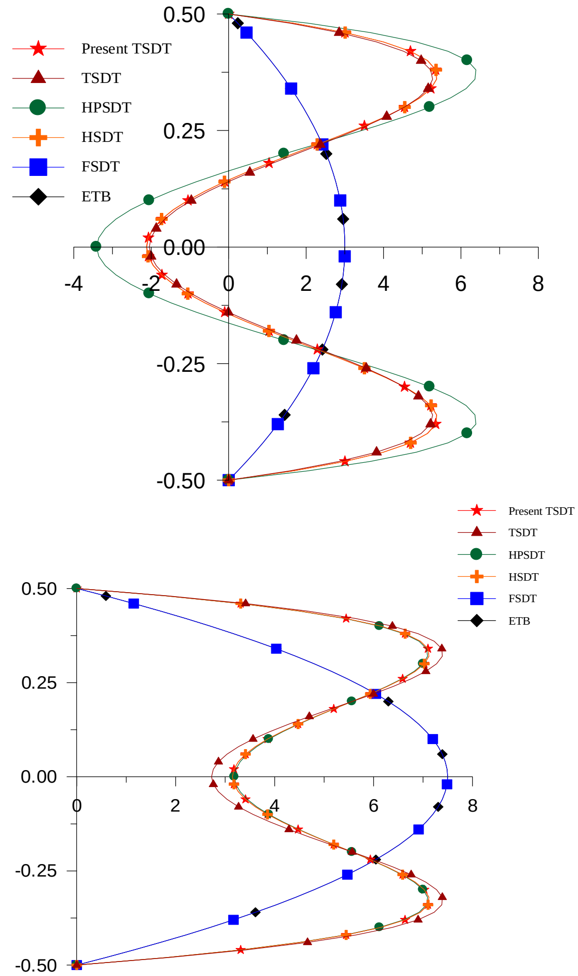

Fig. 6. Variation of transverse shear stress $(\bar{\tau}_{zx})$ through the thickness of cantilever beam at (x = 0.01L, z) obtain using equilibrium equation for aspect ratio 4 and 10.

5 Conclusion

The variationally consistent theoretical formulation of the theory with general solution technique of governing differential equations is presented. The general solutions for beam with uniformly varying load is obtained in case of deep cantilever beam. The displacements and stresses obtained by present theory are in excellent agreement with those of other equivalent refined and higher order theories. The present theory yields the realistic variation of axial displacement and stresses through the thickness of beam. The theory is shown to be capable of predicting the effects of stress concentration on the axial and transverse stresses in the vicinity of the built-in end of the beam which is the region of heavy stress concentration. Thus the validity of the present theory is established.

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